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C-TOTALLY REAL SUBMANIFOLDS

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0. Introduction

C. S. Houh [5], S. T. Yau [10], B. Y. Chen and K. Ogiue [3] have studied totally real submanifolds (anti-holomorphic submanifolds) in an almost Hermitian manifold or a Kählerian manifold of constant holomorphic sectional curvature, and obtained many interesting results.

On the other hand, in the recent paper [8] we have investigated the C-totally real submanifolds in a Sasakian manifold with constant ϕ-holomorphic sectional curvature.

In § 1 we recall some basic formulas for submanifolds in Riemannian manifolds. In § 2 we shall state the fundamental property of C-totally real submanifolds in Sasakian manifolds. In the last section, we investigate C-totally real minimal submanifolds M^n in a constant ϕ-holomorphic sectional curvature and show the pinching theorem for the length of the second fundamental form by using the method of J. Simons [7].

1. Preliminaries

Let $\bar{M}$ be a Riemannian manifold of dimension $n + p$, and M an n-dimensional submanifold of $\bar{M}$. Let $\langle\ ,\ \rangle$ be the metric tensor field on $\bar{M}$ as well as the metric induced on M. We denote by $\bar{\nabla}$ the covariant differentiation in $\bar{M}$, and by ∇ the covariant differentiation in M determined by the induced metric on M. Let $\mathfrak{X}(\bar{M})$ (resp. $\mathfrak{X}(M)$) be the Lie algebra of vector fields on $\bar{M}$ (resp. M), and $\mathfrak{X}^{\perp}(M)$ the set of all vector fields normal to M.

The Gauss-Weingarten formulas are given by

$$\begin{aligned} &\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)\ , \\ &\bar{\nabla}_X N = -A^N(X) + D_X N\ , \quad X, Y \in \mathfrak{X}(M), N \in \mathfrak{X}^{\perp}(M)\ , \end{aligned} \tag{1.1}$$

where D is the connection in the normal bundle. Both A and B are called the second fundamental form of M, and satisfy $\langle A^N(X), Y\rangle = \langle B(X, Y), N\rangle$.

The curvature tensors associated with $\bar{\nabla}, \nabla$ and D are defined by

$$\begin{aligned} \bar{R}(X, Y) &= [\bar{\nabla}_X, \bar{\nabla}_Y] - \bar{\nabla}_{[X,Y]}\ , \\ R(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\ , \\ R^{\perp}(X, Y) &= [D_X, D_Y] - D_{[X,Y]}\ . \end{aligned} \tag{1.2}$$

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If the curvature tensor $R^{\perp}$ of the normal connection D vanishes identically, then the normal connection D is said to be flat.

The Gauss equation is given by

$$
\begin{aligned}
\langle \bar{R}(Z, Y)X, W\rangle &= \langle R(Z, Y)X, W\rangle - \langle B(Y, X), B(Z, W)\rangle \\
&\quad + \langle B(X, Z), B(Y, W)\rangle , \qquad W, X, Y, Z \in \mathfrak{X}(M) .
\end{aligned} \tag{1.3}
$$

Moreover we have the following Ricci equation:

$$
\begin{aligned}
(\bar{R}(Z, Y)N)^{\perp} &= R^{\perp}(Z, Y)N - B(A^N(Y), Z) + B(A^N(Z), Y) , \\
&\qquad Y, Z \in \mathfrak{X}(M),\ N \in \mathfrak{X}^{\perp}(M) ,
\end{aligned} \tag{1.4}
$$

where $(\bar{R}(Z, Y)N)^{\perp}$ is the normal projection of $\bar{R}(Z, Y)N$.

Now we define the covariant derivative of the second fundamental form B as follows:

$$
\tilde{\nabla}_X(B)(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z) \tag{1.5}
$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. For the second fundamental form A we define its covariant derivative by setting

$$
\begin{aligned}
\nabla_X(A)^N(Y) &= \nabla_X(A^N(Y)) - A^{D_X N}(Y) - A^N(\nabla_X Y) , \\
&\qquad X, Y \in \mathfrak{X}(M),\ N \in \mathfrak{X}^{\perp}(M) .
\end{aligned} \tag{1.6}
$$

Clearly we see $\langle \tilde{\nabla}_X(B)(Y, Z), N\rangle = \langle \nabla_X(A)^N(Y), Z\rangle$.

The mean curvature vector H is defined by $H = (1/n)$ trace B. A submanifold M is said to be minimal if $H = 0$ identically. Moreover, M is called a totally geodesic submanifold in $\bar{M}$ if its second fundamental form B is identically zero.

2. *C*-totally real submanifolds

Let $\bar{M}$ be a Sasakian manifold with structure tensors $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. Then the structure tensors satisfy the following equations:

$$
\begin{aligned}
&\phi^2 = -I + \eta \otimes \xi , \quad \phi\xi = 0 , \quad \eta(\phi X) = 0 , \quad \eta(\xi) = 1 , \\
&\bar{\nabla}_{\bar{X}}\xi = \phi\bar{X} , \quad (\bar{\nabla}_{\bar{X}}\phi)\bar{Y} = \eta(\bar{Y})\bar{X} - \langle \bar{X}, \bar{Y}\rangle\xi , \quad \bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M}) .
\end{aligned}
$$

A Sasakian manifold is odd dimensional and orientable. The curvature tensor $\bar{R}(\bar{X}, \bar{Y})$ $(\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M}))$ of a Sasakian manifold $\bar{M}$ satisfies

$$
\begin{aligned}
&\langle \bar{R}(\bar{Z}, \bar{Y})\bar{X}, \bar{W}\rangle - \langle \bar{R}(\bar{Z}, \bar{Y})\phi\bar{X}, \phi\bar{W}\rangle \\
&\quad = \langle Z, W\rangle\langle Y, X\rangle - \langle \bar{Z}, \bar{X}\rangle\langle \bar{Y}, \bar{W}\rangle + \langle \phi\bar{Z}, \bar{X}\rangle\langle \phi\bar{W}, \bar{Y}\rangle \\
&\qquad - \langle \phi\bar{Y}, \bar{X}\rangle\langle \phi\bar{W}, \bar{Z}\rangle
\end{aligned} \tag{2.1}
$$

for any vector fields $\bar{W}, \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\bar{M})$. When the curvature tensor of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$ has the following form

$$\begin{aligned}4\bar{R}(\bar{X},\bar{Y})\bar{Z} = (k+3)\{\langle\bar{Y},\bar{Z}\rangle\bar{X} - \langle\bar{X},\bar{Z}\rangle\bar{Y}\} + (k-1)\{\eta(\bar{X})\eta(\bar{Z})\bar{Y} \\ - \eta(\bar{Y})\eta(\bar{Z})\bar{X} + \langle\bar{X},\bar{Z}\rangle\eta(\bar{Y})\xi - \langle\bar{Y},\bar{Z}\rangle\eta(\bar{X})\xi \\ + \langle\phi\bar{Y},\bar{Z}\rangle\phi\bar{X} + \langle\phi\bar{Z},\bar{X}\rangle\phi\bar{Y} - 2\langle\phi\bar{X},\bar{Y}\rangle\phi\bar{Z}\}\,,\end{aligned} \tag{2.2}$$

then $\bar{M}$ is called a space of constant ϕ-holomorphic sectional curvature. In such a space, k is necessarily constant if $n > 1$.

It is well known that an odd dimensional sphere is Sasakian and a Sasakian manifold is a contact manifold.

Let us recall the definition of a C-totally real submanifold in a Sasakian manifold. Let $\bar{M}$ be a $(2n+1)$-dimensional contact manifold with contact form η. The Pfaffian equation $\eta = 0$ determines in $\bar{M}$ a $2n$-dimensional distribution, which is called the contact distribution [6]. A submanifold M in $\bar{M}$ is said to be an integral submanifold of the contact distribution if and only if every tangent vector of M belongs to the contact distribution. We shall call the integral submanifold M of the contact distribution of a Sasakian manifold a C-totally real submanifold. Then we have known $\dim M \le n$, and the following theorem has been proved [8]:

Theorem A. *Let M be an m $(m \le n)$ dimensional C-totally real submanifold in a Sasakian manifold $\bar{M}^{2n+1}$ with structure tensors $(\phi, \xi, \eta, \langle\ ,\ \rangle)$. Then we have the following.*

(i) *The second fundamental form of ξ direction is identically zero.*

(ii) *If $X \in \mathfrak{X}(M)$, then $\phi X \in \mathfrak{X}^{\perp}(M)$.*

(iii) *If $m = n$, then $A^{\phi X}(Y) = A^{\phi Y}(X)$, $X, Y \in \mathfrak{X}(M)$.*

Making use of Theorem A, (1.3) and (2.2) we can easily prove

Proposition 2.1. *Let M be an m $(\le n)$-dimensional C-totally real submanifold of* a $(2n+1)$*-dimensional Sasakian manifold $\bar{M}^{2n+1}$ with constant ϕ-holomorphic sectional curvature k. If M is totally geodesic, then M is of constant curvature $\frac{1}{4}(k+3)$.*

In the following, we deal with an n-dimensional C-totally real submanifold M of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}^{2n+1}$. We shall show

Theorem 2.2. *Let M be an n-dimensional C-totally real submanifold of a Sasakian manifold $\bar{M}^{2n+1}$. Then the normal connection is flat if and only if the submanifold M is of constant curvature* 1.

Proof. Using (1.4) and taking account of Theorem A (iii) we can obtain

$$\begin{aligned}\langle\bar{R}(Z,Y)\phi X, \phi W\rangle = \langle R^{\perp}(Z,Y)\phi X, \phi W\rangle - \langle B(X,Y), B(W,Z)\rangle \\ + \langle B(X,Z), B(W,Y)\rangle\,. \qquad W, X, Y, Z \in \mathfrak{X}(M)\,,\end{aligned}$$

which together with (1.3) implies

$$\bar{R}(Z,Y)\phi X - \phi\bar{R}(Z,Y)X + \phi R(Z,Y)X = R^{\perp}(Z,Y)\phi X\,.$$

Consequently, regarding to (2.1) we get

$$\langle R(Z,Y)X,W\rangle - \langle Z,W\rangle\langle Y,X\rangle + \langle Z,X\rangle\langle Y,W\rangle = \langle R^{\perp}(Z,Y)\phi X, \phi W\rangle ,$$

which completes the proof because of

$$\langle \bar{R}(Z,Y)N,\xi\rangle = \eta(Z)\langle Y,N\rangle - \eta(Y)\langle X,N\rangle = 0 .$$

Theorem 2.3. *Let M be an n-dimensional C-totally real submanifold in $\bar{M}^{2n+1}$. If the second fundamental form of M is parallel, then M is totally geodesic.*

Proof. Let $X, Y, Z \in \mathfrak{X}(M)$. By (1.5) we have

$$\langle B(X,Y),\phi Z\rangle = -\langle \tilde{\nabla}_Z(B)(X,Y),\xi\rangle = 0 ,$$

which shows that M is totally geodesic.

3. C-totally real minimal submanifolds

In this section we assume that $\bar{M}^{2n+1}(k)$ is a $(2n+1)$-dimensional Sasakian manifold with constant ϕ-holomorphic sectional curvature k, and M is an n-dimensional C-totally real submanifold of $\bar{M}^{2n+1}(k)$. Then the Simons' type formula for the second fundamental form A is given by

$$\nabla^2 A = -A\circ\tilde{A} - \underset{\sim}{A}\circ A + \tfrac{1}{4}\{(n+1)k + 3n - 1\}A ,$$

where the operators $\tilde{A}$ and $\underset{\sim}{A}$ are defined by

$$\tilde{A} = {}^tA\circ A , \qquad \underset{\sim}{A} = \sum_{\alpha=n+1}^{2n+1} (\mathrm{ad}\, A^{\alpha})\mathrm{ad}\, A^{\alpha} .$$

Now we take a frame $E_1, \cdots, E_n$ for $T_P(M)$ and a frame $\phi E_1, \cdots, \phi E_n, \xi$ for $T_P(M)^{\perp}$, and for simplicity write A^{i^*} for $A^{\phi E_i}$. As $A^{\xi} = 0$, we have $\underset{\sim}{A} = \sum_{i=1}^{n} (\mathrm{ad}\, A^{i^*})\mathrm{ad}\, A^{i^*}$. By the method of Simons we can easily derive the inequality:

$$\langle A\circ A, A\rangle + \langle \underset{\sim}{A}\circ A, A\rangle \leq \left(2 - \frac{1}{n}\right)\|A\|^4 .$$

If M is compact, then

$$\begin{aligned} &\int_M \{\langle\nabla A,\nabla A\rangle - \|A\|^2\} \\ (3.1) \qquad &\leq \int_M \left\{\left(2-\frac{1}{n}\right)\|A\|^2 - \frac{1}{4}(n+1)(k+3)\right\}\|A\|^2 . \end{aligned}$$

Next we shall prove that the left hand side of (3.1) is nonnegative at each point of M. Owing to (1.6) we have

$$\begin{aligned}\langle \nabla A, \nabla A\rangle &= \sum_{i,j,k=1}^{n} \langle \nabla_{E_i}(A)^{j^*}(E_k), \nabla_{E_i}(A)^{j^*}(E_k)\rangle \\ &\quad + \sum_{i,j=1}^{n} \langle \nabla_{E_i}(A)^{\xi}(E_j), \nabla_{E_i}(A)^{\xi}(E_j)\rangle \\ &= \sum_{i,j,k} \langle \nabla_{E_i}(A)^{j^*}(E_k), \nabla_{E_i}(A)^{j^*}(E_k)\rangle + \|A\|^2 ,\end{aligned}$$

which implies $\langle \nabla A, \nabla A\rangle - \|A\|^2 \geq 0$. Hence

$$0 \leq \int_M \left\{\left(2 - \frac{1}{n}\right)\|A\|^2 - \frac{1}{4}(n+1)(k+3)\right\}\|A\|^2 . \tag{3.2}$$

Therefore we obtain

Theorem 3.1. *Let $\overline{M}^{2n+1}(k)$ be a $(2n+1)$-dimensional Sasakian manifold with constant ϕ-holomorphic sectional curvature k, and M a compact n-dimensional C-totally real minimal submanifold of $\overline{M}^{2n+1}(k)$. If*

$$\|A\|^2 < \tfrac{1}{4}n(n+1)(k+3)/(2n-1) ,$$

or equivalently

$$\rho > \tfrac{1}{2}n^2(n-2)(k+3)/(2n-1) ,$$

then M is totally geodesic, where ρ is the scalar curvature of M.

Theorem 3.2. *Let M be an n-dimensional C-totally real minimal submanifold of $\overline{M}^{2n+1}(k)$. If the sectional curvature of M is constant, say C, then either $C = \frac{1}{4}(k+3)$ (i.e., M is totally geodesic) or $C \leq 0$.*

Proof. We calculate $\langle A \circ \tilde{A}, A\rangle$ and $\langle \underline{A} \circ A, A\rangle$ in the following ways. In the first place, by virtue of (1.3) and (2.2) we have

$$\begin{aligned}\langle A \circ \tilde{A}, A\rangle &= \sum_{i,j} (\operatorname{trace} A^{i^*}A^{j^*})^2 = \operatorname{trace}\left(\sum_i (A^{i^*})^2\right)^2 \\ &= (n-1)(\tfrac{1}{4}(k+3) - C)\|A\|^2 .\end{aligned} \tag{3.3}$$

On the other hand, using (1.3) we get

$$-(\tfrac{1}{4}(k+3) - C)\|A\|^2 = \sum_{k,t} \operatorname{trace} A^{k^*}A^{t^*}A^{k^*}A^{t^*} - \langle A \circ \tilde{A}, A\rangle . \tag{3.4}$$

In the next place, from the definition of A it follows that

$$\langle \underline{A} \circ A, A\rangle = 2\sum_{k,t} \operatorname{trace}(A^{t^*})^2(A^{k^*})^2 - 2\sum_{k,t} \operatorname{trace} A^{k^*}A^{t^*}A^{k^*}A^{t^*} . \tag{3.5}$$

Therefore by virtue of (3.3), (3.4) and (3.5) we obtain

$$\langle \underline{A} \circ A, A\rangle = 2(\tfrac{1}{4}(k+3) - C)\|A\|^2 , \tag{3.6}$$

which means

$$\langle \nabla A, \nabla A \rangle - \|A\|^2 = n(n^2 - 1)C(C - \tfrac{1}{4}(k + 3)) . \tag{3.7}$$

This completes our assertion.

The following result is an immediate consequence of (3.6).

Theorem 3.3. *Let M be an n-dimensional C-totally real minimal submanifold in $\bar{M}^{2n+1}(k)$. If the sectional curvature of M is constant, and $\langle \nabla A, \nabla A \rangle = \|A\|^2$ holds, then M is either totally geodesic or flat.*

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